Mellor-Yamada 2.5 level Turbulence Model

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Outline

- A brief history
- Details about the model
- Application by ICOM
- Summaries

A Brief History

- Modelled Reynolds Stress and Fluxes equations. (1973, G. Mellor, Analytic Prediction of the Properties of Stratified Planetary Surface Layers, *J. Atmos. Sci.)*
- Made different level of simplification and Classified them into Level 4, 3, 2, 1,(1974, G. Mellor and T. Yamada, A Hierarchy of Turbulence Closure Models for Planetary Boundary Layers, *J. Atmos. Sci.*)
- Level 2.5 model,(1982, G. Mellor and T. Yamada, Development of a Turbulence Closure Model for Geophysical Fluid Problems. *Rev. Geophy. and Space Phy.)*
- Removal of a slight inconsistency of the former scaling equilibrium model, so simplified the turbulent exchange coefficients and eliminated the dependence on shearing rates of realizability conditions. (1988, B. Galperin, et al. , A quasi-equilibrium Turbulent Energy Model for Geophysical Flows. *J. Atmos. Sci.*)

Reynolds Equations

$$\frac{\partial U_i}{\partial x_i} = 0$$

$$\frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x_k}\left(U_k U_j - \overline{u_k u_j}\right) + \varepsilon_{jkl} f_k U_l = -\frac{\partial P}{\partial x_j} - g_j \beta \Theta + \nu \nabla^2 U_j$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial x_k}\left(U_k \Theta - \overline{u_k \theta}\right) = \alpha \nabla^2 \Theta$$

Where, *U*, *P* and *Θ* are mean velocity, pressure and potential temperature. The low case terms, u,θ are fluctuating components. *v*, *α* are kinematic viscosity and heat conductivity respectively. *β* is the coefficient of thermal expansion,

$$\beta = -\left(\partial \rho / \partial T\right)_p / \rho$$

Fluctuation Equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_k}\left(U_k u_j + U_j u_k + u_k u_j - \overline{u_k u_j}\right) + \varepsilon_{jkl} f_k u_l = -\frac{\partial p}{\partial x_j} - g_j \beta\theta + \nu\nabla^2 u_j$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial x_k}\left(U_k \theta + u_k \Theta + u_k \theta - \overline{u_k \theta}\right) = \alpha\nabla^2\theta$$

From the above 3 equations, we can get the Reynolds stresses and fluxes equations which forms the basic of the turbulence modelling

Reynolds Stresses Equation

$$
\begin{aligned}
\frac{D\overline{u_i u_j}}{Dt} + f_k\left(\varepsilon_{jkl}\overline{u_l u_i} + \varepsilon_{ikl}\overline{u_l u_j}\right) = & -\left(\overline{u_k u_i}\frac{\partial U_j}{\partial x_k} + \overline{u_k u_j}\frac{\partial U_i}{\partial x_k}\right) \\
& -\beta\left(g_j\overline{u_i\theta} + g_i\overline{u_j\theta}\right) \\
& +\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \\
& -2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \\
& -\frac{\partial}{\partial x_k}\left(\overline{u_k u_i u_j} - \nu\frac{\partial}{\partial x_k}\overline{u_i u_j}\right) \\
& -\left(\frac{\partial}{\partial x_j}\overline{p u_i} + \frac{\partial}{\partial x_i}\overline{p u_j}\right)
\end{aligned}
$$

Reynolds Fluxes and θ^2 Equation

$$\frac{D\overline{u_j\theta}}{Dt} + f_k\varepsilon_{jkl}\overline{u_l\theta} = -\left(\overline{u_j u_k}\frac{\partial\Theta}{\partial x_k} + \overline{\theta u_k}\frac{\partial U_j}{\partial x_k}\right)$$

$$-\beta g_j\overline{\theta^2}$$

$$+\overline{p\frac{\partial\theta}{\partial x_j}}$$

$$-(\alpha+\nu)\overline{\frac{\partial u_j}{\partial x_k}\frac{\partial\theta}{\partial x_k}}$$

$$-\frac{\partial}{\partial x_k}\left(\overline{u_k u_j\theta} - \alpha\overline{u_j\frac{\partial\theta}{\partial x_k}} - \nu\overline{\theta\frac{\partial u_j}{\partial x_k}}\right)$$

$$-\frac{\partial}{\partial x_j}\overline{p\theta}$$

$$\frac{D\overline{\theta^2}}{Dt} = -\frac{\partial}{\partial x_k}\left(\overline{u_k\theta^2} - \alpha\frac{\partial\overline{\theta^2}}{\partial x_k}\right) - 2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k} - 2\alpha\overline{\frac{\partial\theta}{\partial x_k}\frac{\partial\theta}{\partial x_k}}$$

Modelling assumptions for Reynolds Strees and fluxes equations

$$\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}=-\frac{q}{3l_1}\left(\overline{u_iu_j}-\frac{\delta_{ij}}{3}q^2\right)+Cq^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) \qquad \overline{p\frac{\partial \theta}{\partial x_j}}=-\frac{q}{3l_2}\overline{u_j\theta}$$

$$2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}=\frac{2}{3}\frac{q^3}{\Lambda_1}\delta_{ij} \qquad (\alpha+\nu)\overline{\frac{\partial u_j}{\partial x_k}\frac{\partial \theta}{\partial x_k}}=0 \qquad 2\alpha\overline{\frac{\partial \theta}{\partial x_k}\frac{\partial \theta}{\partial x_k}}=2\frac{q}{\Lambda_2}\overline{\theta^2}$$

$$\overline{u_ku_iu_j}=-\frac{3}{5}qlS_q\left(\frac{\partial\overline{u_iu_j}}{\partial x_k}+\frac{\partial\overline{u_iu_k}}{\partial x_j}+\frac{\partial\overline{u_ju_k}}{\partial x_i}\right)$$

$$\overline{u_ku_j\theta}=-qlS_{u\theta}\left(\frac{\partial\overline{u_k\theta}}{\partial x_j}+\frac{\partial\overline{u_j\theta}}{\partial x_k}\right) \qquad \overline{u_k\theta^2}=-qlS_\theta\frac{\partial\overline{\theta^2}}{\partial x_k} \qquad \overline{pu_i}=\overline{p\theta}=0$$

Parameters and Coefficients

l_1, l_2, C Rotta's energy redistribution hypothesis

Λ_1, Λ_2 Kolmogoroff local isotropy hypothesis

$S_q, S_{u\theta}, S_\theta$ Dimensionless numbers

$$l_1 = A_1 l, l_2 = A_2 l, \Lambda_1 = B_1 l, \Lambda_2 = B_2 l$$

$$(A_1, A_2, B_1, B_2, C) = (0.92, 0.74, 16.6, 10.1, 0.08)$$

Level 4 Model

$$\frac{D\overline{u_i u_j}}{Dt} + f_k\left(\varepsilon_{jkl}\overline{u_l u_i} + \varepsilon_{ikl}\overline{u_l u_j}\right) = \frac{\partial}{\partial x_k}\left[\frac{3}{5} lqS_q\left(\frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\partial \overline{u_i u_k}}{\partial x_j} + \frac{\partial \overline{u_j u_k}}{\partial x_i}\right)\right]$$

$$-\left(\overline{u_k u_i}\frac{\partial U_j}{\partial x_k} + \overline{u_k u_j}\frac{\partial U_i}{\partial x_k}\right) - \beta\left(g_j\overline{u_i\theta} + g_i\overline{u_j\theta}\right)$$

$$-\frac{q}{3l_1}\left(\overline{u_i u_j} - \frac{\delta_{ij}}{3}q^2\right) + Cq^2\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\frac{q^2}{\Lambda_1}\delta_{ij}$$

$$\frac{D\overline{u_j\theta}}{Dt} + f_k\varepsilon_{jkl}\overline{u_l\theta} = \frac{\partial}{\partial x_k}\left[lqS_{u\theta}\left(\frac{\partial\overline{u_j\theta}}{\partial x_k} + \frac{\partial\overline{u_k\theta}}{\partial x_j}\right)\right]$$

$$-\left(\overline{u_j u_k}\frac{\partial \Theta}{\partial x_k} + \overline{\theta u_k}\frac{\partial U_j}{\partial x_k}\right) - \beta g_j\overline{\theta^2} - \frac{q}{3l_2}\overline{u_j\theta}$$

$$\frac{D\overline{\theta^2}}{Dt} = -\frac{\partial}{\partial x_k}\left(lqS_\theta\frac{\partial\overline{\theta^2}}{\partial x_k} + \alpha\frac{\overline{\partial\theta^2}}{\partial x_k}\right) - 2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k} - 2\frac{q}{\Lambda_2}\overline{\theta^2}$$

Further Simplification to Level 2.5 Model (Galperin Version)

By contracting Reynolds stress equation we can get turbulence kinetic energy equation

$$\frac{Dq^2}{Dt} = \ldots$$

Subtract (above equation)*δ_{ij} /3 from Reynolds stress equation, we can get another equation

$$\frac{D}{D}\left(u_i u_j - \frac{\delta_{ij}}{3} q^2 \right) = \ldots$$

Here, $q^2 = \overline{u_i u_i}$

Introduce non-dimensional tensors a_{ij}, b_j to characterizing the departure from local isotropy

$$\overline{u_i u_j} = \left(\frac{\delta_{ij}}{3} + a_{ij} \right) q^2, \quad a_{ii} = 0$$

$$\overline{u_j \theta} = b_j q \phi$$

where $\phi^2 = \overline{\theta^2}, \quad a = \left\| a_{ij} \right\|, \quad b = \left\| b_j \right\|, \quad O(a) = O(b)$

Do following scaling:

$$U_x^2 = O\left(\left\| \frac{\partial U_i}{\partial x_j} \right\|^2 \right), \quad \Theta_x^2 = O\left(\left\| \frac{\partial \Theta}{\partial x_j} \right\|^2 \right), \quad l = O(l_1) = O(l_2)$$

$$a^2 = l / \Lambda, \quad b^2 = l / \Lambda, \quad U_x = a^{-1} q / \Lambda, \quad \Theta_x = b^{-1} \phi / \Lambda \qquad g\beta\phi = b^{-1} q^2 / \Lambda$$

and assume $\quad Uq^2 / L = aq^3 / \Lambda$

Level 2.5 Model

By neglecting terms of $O(a^2)$, we get Galperin version of Level 2.5 Model

$$\frac{Dq^2}{Dt} - \frac{\partial}{\partial x_k}\left(lqS_q \frac{\partial q^2}{\partial x_k} \right) = -2\overline{u_k u_l}\frac{\partial U_k}{\partial x_l} - 2\beta g_k \overline{u_k \theta} - 2\frac{q^3}{\Lambda_1} + O\left(a^2\right)$$

$$\overline{u_i u_j} = \frac{1}{3}\delta_{ij} q^2 - \frac{3l_1}{q}\left[\left(\overline{u_k u_l} - Cq^2 \delta_{ki} \right) \frac{\partial U_j}{\partial x_k} + \left(\overline{u_k u_j} - Cq^2 \delta_{kj} \right) \frac{\partial U_i}{\partial x_k} \right]$$

$$- \frac{3l_1}{q}\beta\left(g_j \overline{u_i \theta} + g_i \overline{u_j \theta} \right) - \frac{l_1}{q}\delta_{ij}\frac{2q^3}{\Lambda_1} + f_k\left(\varepsilon_{jkl}\overline{u_l u_i} + \varepsilon_{ikl}\overline{u_l u_j} \right) + O\left(a^2\right)$$

Do similar work on turbulence fluxes equation,

$$\overline{u_j \theta} = -\frac{3l_2}{q}\left[\overline{u_j u_k}\frac{\partial \Theta}{\partial x_k} + \overline{u_k \theta}\frac{\partial U_j}{\partial x_k} + \beta g_j \overline{\theta\theta} + f_k \varepsilon_{jkl}\overline{u_l \theta} \right]$$

$$\boxed{\cancel{\frac{D\overline{\theta^2}}{Dt} - \frac{\partial}{\partial x_k}\left(lqS_\theta \frac{\partial \overline{\theta^2}}{\partial x_k} \right)} = -2\overline{u_k \theta}\frac{\partial \Theta}{\partial x_k} - \frac{2q}{\Lambda_2}\overline{\theta^2}} \quad \Rightarrow \quad \theta^2 = -\frac{\Lambda_2}{q} u_k \theta \frac{\partial \Theta}{\partial x_k}$$

The Boundary Layer Approximation for Level 2.5 Model

(1). neglecting Coriolis term in turbulence moment equation

(2). neglecting $\dfrac{\partial U_j}{\partial x_k}$ except $\dfrac{\partial U_j}{\partial z}$

(3). $g_i = (0,0,-g)$

$$-uw = K_M \frac{\partial U}{\partial z}, \quad -vw = K_M \frac{\partial V}{\partial z}, \quad -\theta w = K_H \frac{\partial \Theta}{\partial z}$$

$$K_M = lqS_M, \quad K_H = lqS_H$$

$$G_M = \frac{l^2}{q^2}\left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right]$$

$$G_H = -\frac{l^2}{q^2}\beta g \frac{\partial \Theta}{\partial z}$$

Using $\overline{u_i u_j} = \ldots$ and $\overline{u_i \theta} = \ldots$

$$S_M = A_1 \frac{1 - 3C - \left(6A_1 / B_1\right) - 3A_2 G_H \left(\left(B_2 - 3A_2\right)\left(1 - \left(6A_1 / B_1\right)\right) - 3C\left(B_2 + 6A_1\right)\right)}{\left(1 - 3A_2 G_H \left(6A_1 + B_2\right)\right)\left(1 - 9A_1 A_2 G_H\right)}$$

$$S_H = A_2 \frac{1 - \left(6A_1 / B_1\right)}{1 - 3A_2 G_H \left(6A_1 + B_2\right)}$$

$$K_M = lqS_M, \quad K_H = lqS_H$$

So far, the 2.5 level boundary layer turbulence model looks finished, but where is the equation for l

q^2l equation

q2l's equation is based on the integral of the two point correlation function.

$$q^2 l(\vec{x},t) \sim \int u_i(\vec{x},t) u_i(\vec{x}+\vec{r},t) dr$$

The exact equations for q^2l can be derived (Rotta, 1951), Mellor and Herring's simplified form is

$$\frac{D}{Dt}\left(q^2 l\right) - \frac{\partial}{\partial z}\left[q l S_l \frac{\partial}{\partial z}\left(q^2 l\right)\right] = l E_1 \left[P_s + P_b\right] - \frac{q^3}{B_1}\left[1 + E_2\left(\frac{l}{\kappa L}\right)^2\right]$$

where $S_l = 0.2, \quad E_1 = 1.8 \quad E_2 = 1.33$

$\kappa = 0.40$ is Von Karman coefficient

L is suppose to be a measure of the distance away from the wall

$$L^{-1}(r) = \frac{1}{2\pi} \iint \frac{dA(r_0)}{(r - r_0)^3}$$

Boundary Conditions

When approach bottom or top boundary, buoyancy production terms vanish, assuming production equals dissipation,

$$q^3 / \Lambda_1 = -\overline{uw} \partial U / \partial z$$
$$-uw = \tau$$
$$\partial U / \partial z = \tau^{1/2} / l$$

$$\Longrightarrow \quad q^3 = B_1 \tau^{3/2} \quad \Longrightarrow \quad q^2 = B_1^{2/3} \tau$$

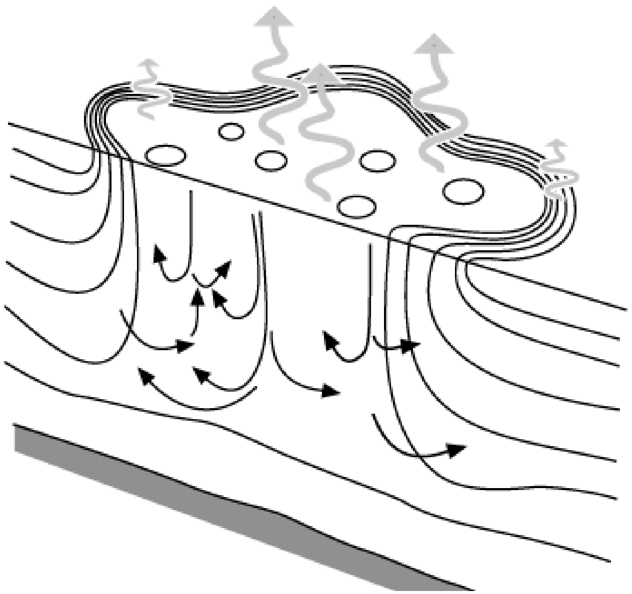
On solid surface $q^2 l = 0$

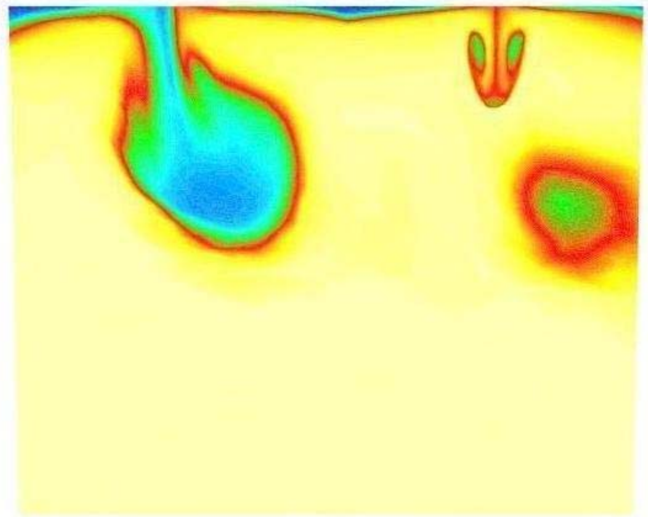
On free stream boundary, a small value should be added to

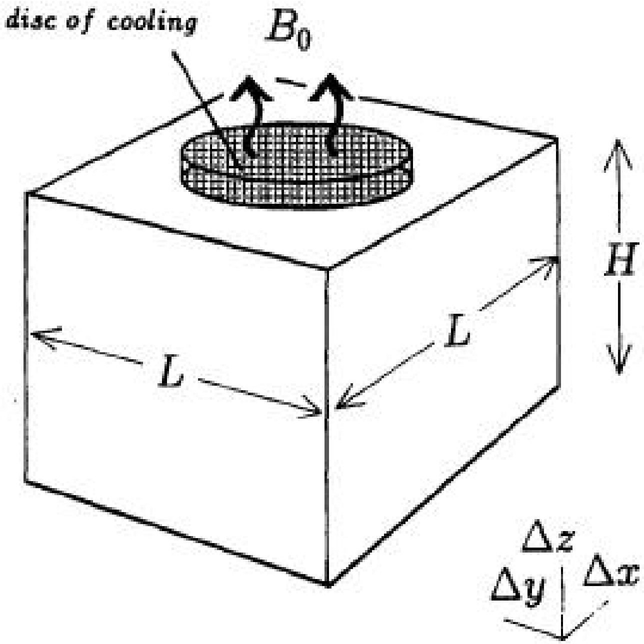
Mellor claimed: Solutions are quite insensitive to free stream values of $q^2 l$

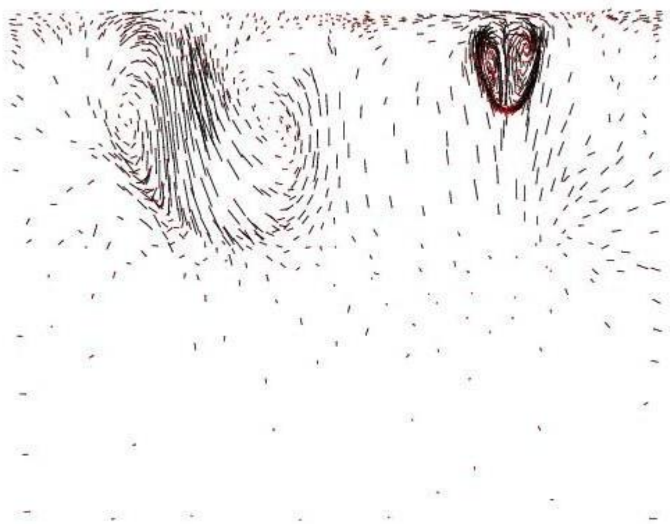
Application of M-Y2.5 in ICOM

Ocean Deep Convection

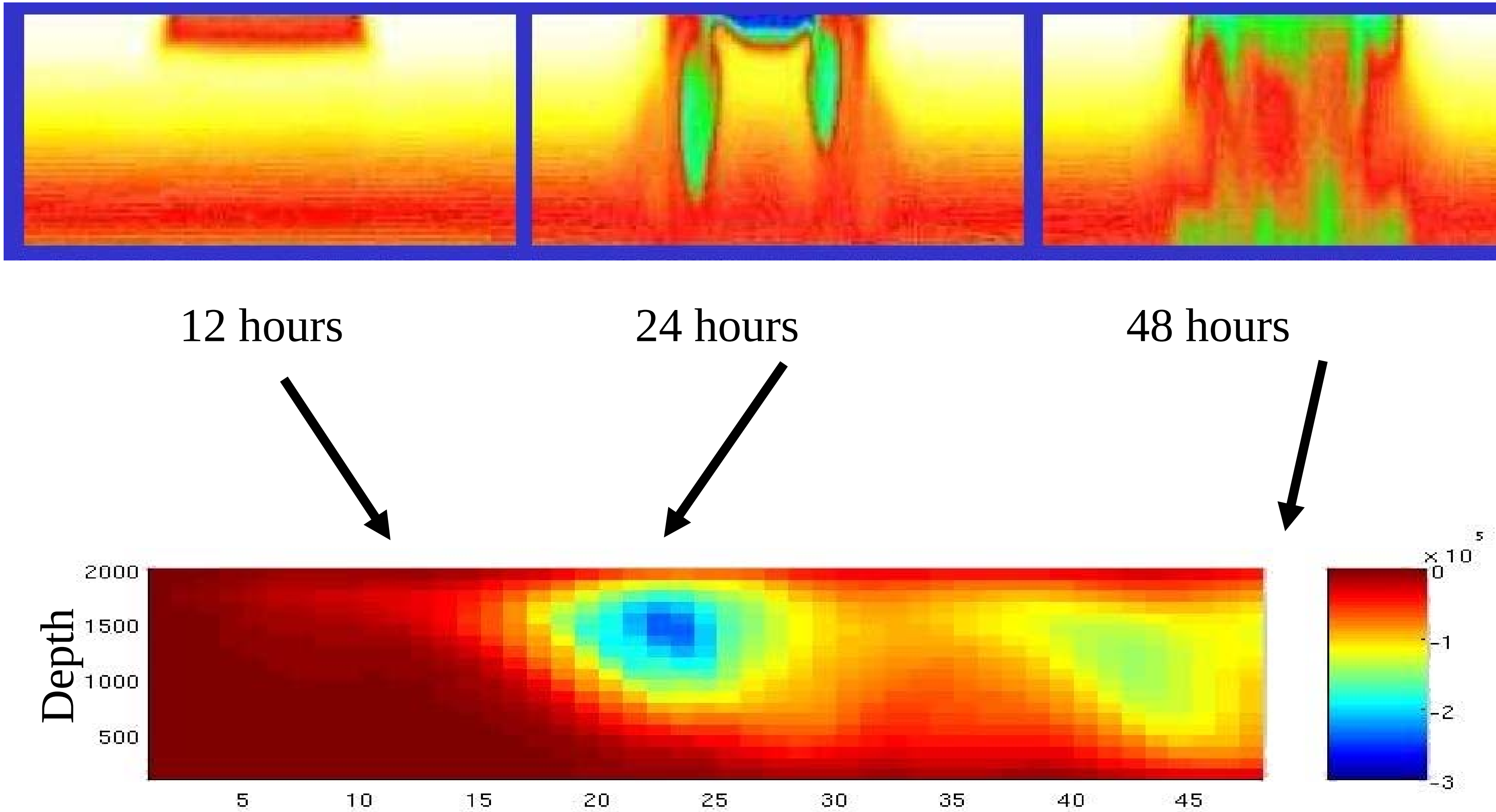






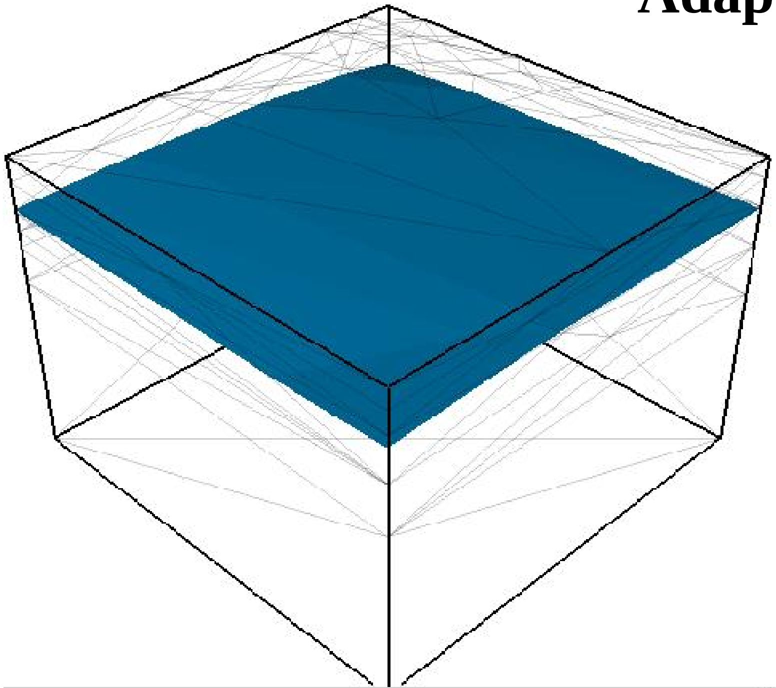


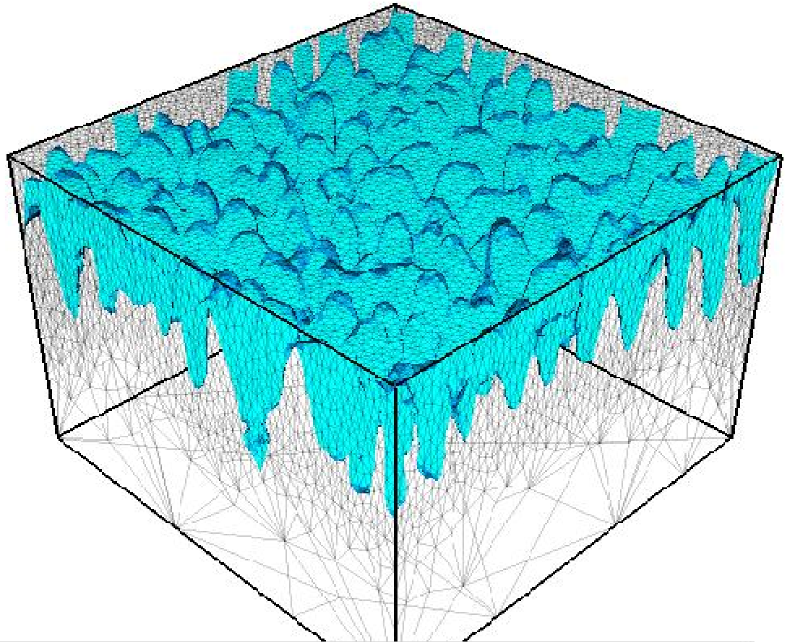
Vertical Heat Flux

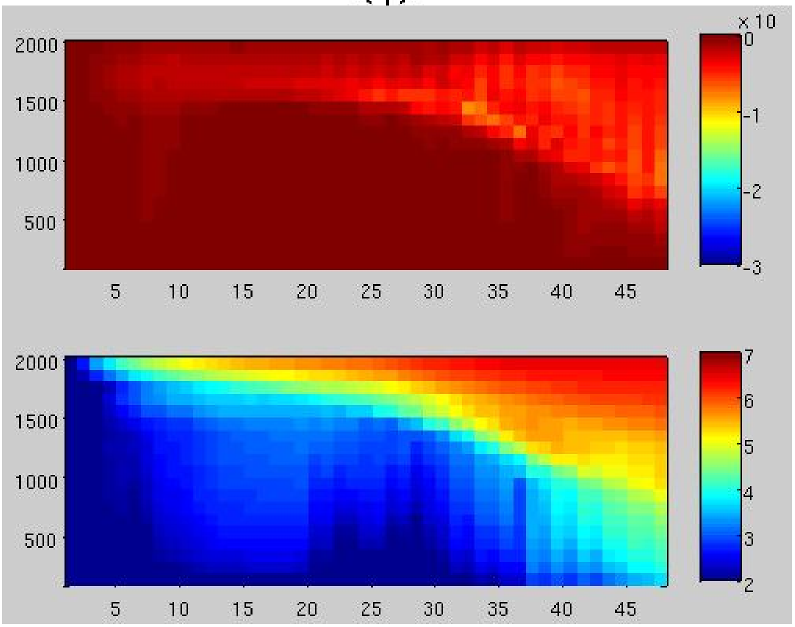


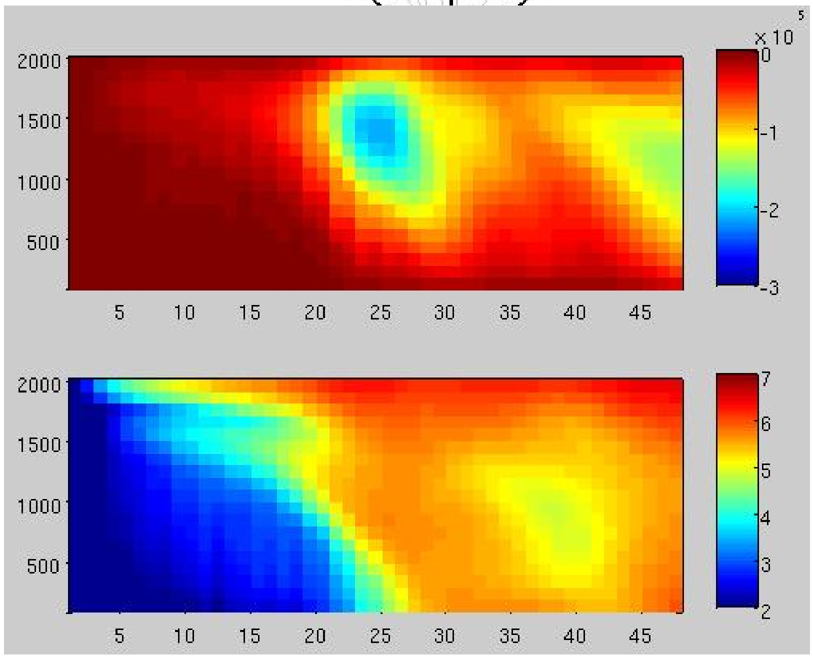
Variation in vertical heat flux with time / hours

Adaptive results









Summaries

- Mellor-Yamada 2.5 level turbulence model is a 2-equation 2nd order moment model. Its complexity and robustness is proper for a wide variety of engineering and geophysical flow simulation
- It has a set of simple and consistent expression of parameters when applied to PBL problem
- It has shown it abilities to simulate the cases of stable and unstable stratification.
- All length scales are proportional to each other everywhere
- Isotropic dissipation is not valid near boundary
- Turbulence scale boundary condition need to be improved at free surface
- Turbulence scale equation